Functions

Suppose A and B are sets. A function f from A to B is a rule that assigns to every element $a \in A$ an element $f(a) \in B$.

image of a under f

(Notation: $f : A \to B$)

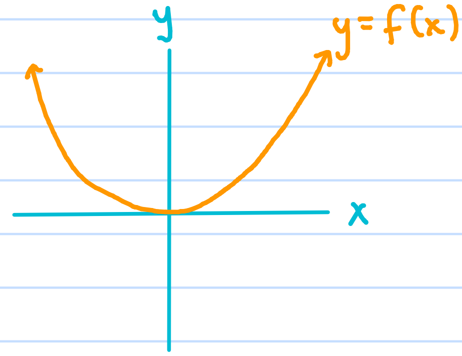
domain codomain

- The range of f is the set
  $$f(A) = \{ f(a) : a \in A \} \subseteq B.$$
- f is called injective (one-to-one) if every element in its range is the image of exactly one point in its domain.

  Equivalently: f is injective if, whenever $f(a) = f(a')$, we have $a = a'$
- f is called surjective (onto) if $f(A) = B$.

  Equivalently: f is surjective if
  $$\forall b \in B, \ \exists a \in A \text{ s.t. } f(a) = b$$
- f is called bijective if it is both injective and surjective.

1a) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$



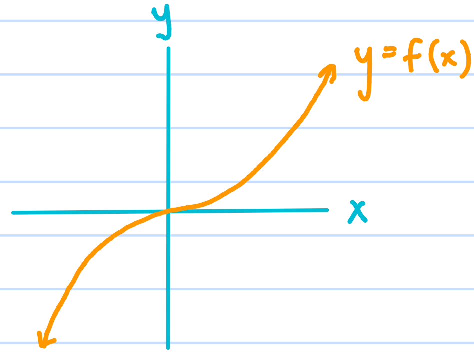
- f is not injective:

  ex: $f(-1) = f(1)$.

- f is not surjective:

  ex: there is no $x \in \mathbb{R}$ with $f(x) = -1$.

1b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$



- f is injective:

  If $x, x' \in \mathbb{R}$, $x^3 = (x')^3$,

  then $x = x'$.

- f is surjective:

  $\forall y \in \mathbb{R}$, $\exists x \in \mathbb{R}$ s.t. $x^3 = y$.

1c) $f: [0, \infty) \to [0, \infty)$, $f(x) = x^2$.

- f is injective:

  If $x, x' \in [0, \infty)$, $x^2 = (x')^2$

  then $x = x'$.

- f is surjective:

  $\forall y \in [0, \infty)$, $\exists x \in [0, \infty)$ s.t. $x^2 = y$.

Exs:

2a) $f : \underset{\text{domain}}{\mathbb{N}} \to \underset{\text{codomain}}{\mathbb{Z}}$, $\quad f(n) = n-1$.

- f is injective:

  Suppose $f(n) = f(m)$, for $m, n \in \mathbb{N}$.

  Then $n-1 = m-1 \implies n = m$.

- f is not surjective:

  $$f(\mathbb{N}) = \{f(n) : n \in \mathbb{N}\}$$
  $$= \{n-1 : n \in \{1,2,3,\dots\}\}$$
  $$= \{0,1,2,\dots\} = \mathbb{Z}_{\geq 0} \neq \mathbb{Z}.$$

2b) $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$, $\quad f(n) = |n|$.

- f is not injective:

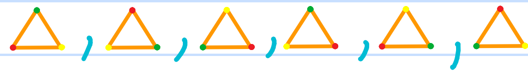
  ex: $-1, 1 \in \mathbb{Z}$, $-1 \neq 1$, but

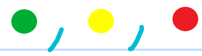
  $$f(-1) = |-1| = |1| = f(1).$$

- f is surjective:

  $\forall n \in \mathbb{Z}_{\geq 0}$, we have that $n \in \mathbb{Z}$ $\quad (\mathbb{Z}_{\geq 0} \subseteq \mathbb{Z})$

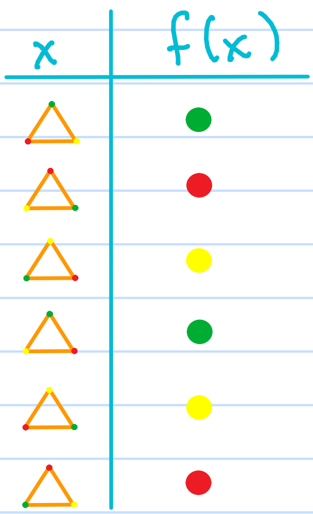
  and that $f(n) = |n| = n$.

3 a) A = {  }

B = {  }

f : A → B defined by

f(triangle) = ball with same color as top vertex of triangle



- f is <u>not</u> injective
- f is surjective

3b) Let X be a set and, $\forall A \subseteq X$, define

$f_A : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$f_A(B) = A \setminus B, \quad \forall B \in \mathcal{P}(X).$$

For what choices of A will f_A be:

- injective? (note: $f_A(B) = f_A(C) \iff (A \cap B) = (A \cap C)$)

...thoughts go here...

- If $A \neq X$ then $\exists x \in X \setminus A$.

  Then $f_A(\emptyset) = A \setminus \emptyset = A$, and

  $f_A(\{x\}) = A \setminus \{x\} = A$,

  Since $\emptyset \neq \{x\}$, f_A is not injective.

- If $A = X$ then, $\forall B \in \mathcal{P}(X)$,

  $f_A(B) = A \setminus B = X \setminus B = B^c$.

  Suppose $B, C \in \mathcal{P}(X)$ and $f_A(B) = f_A(C)$.

  Then $B^c = C^c \Rightarrow (B^c)^c = (C^c)^c$

  $\Rightarrow B = C$.

  Therefore f_A is injective.

Conclusion: f_A will be injective if and only if $A = X$.

- surjective? (note: $\forall B \in \mathcal{P}(X)$, $f_A(B) \subseteq A$)
  - If $A \neq X$ then $\exists x \in X \setminus A$. Since $x \notin A$, we have that $x \notin A \setminus B$, $\forall B \in \mathcal{P}(X)$. Therefore $\{x\} \notin f_A(\mathcal{P}(X))$, so f_A is not surjective. (range of f_A)
  - If $A = X$ then, $\forall B \in \mathcal{P}(X)$, we have that $B^c \in \mathcal{P}(X)$ and

$$f_A(B^c) = A \setminus (B^c) = X \setminus (B^c) = (B^c)^c = B.$$

Therefore f_A is surjective.

Conclusion: f_A will be surjective if and only if $A = X$.

Important facts:

- If $f : A \to B$ is a bijection then $|A| = |B|$.
- If $|A| < \infty$, $|B| < \infty$, and $f : A \to B$ is a function then f is injective if and only if f is surjective.

A few familiar definitions:

- If $f : A \to B$ is a bijection then the <u>inverse function</u> $f^{-1} : B \to A$ is defined by the rule that, $\forall y \in B$,

$$f^{-1}(y) = x \iff f(x) = y .$$

Note: f being a bijection guarantees that
  - f^{-1} is well-defined
  - f^{-1} is a bijection
- If $f : A \to B$ and $g : B \to C$ then the <u>composite function</u> $g \circ f : A \to C$ is defined by $(g \circ f)(x) = g(f(x))$, $\forall x \in A$.

  Note: $(g \circ f)(A) = g(f(A))$,

  but $g(f(A)) \neq g(B)$, in general.

Finally:

Generalized Cartesian products:

The Cartesian product of a collection of sets $\{A_i\}_{i \in I}$ is defined by

$$\prod_{i \in I} A_i = \left\{ f : I \to \bigcup_{i \in I} A_i \ : \ \forall i \in I,\ f(i) \in A_i \right\}.$$

Exs:

1) $I = \{1, 2\}$, $A_1 = A$, $A_2 = B$.

$$\prod_{i \in \{1,2\}} A_i = \{ f : \{1,2\} \to A \cup B : f(1) \in A,\ f(2) \in B \}$$

cf: $A \times B = \{ (a,b) : a \in A,\ b \in B \}$

2) $I = \mathbb{N}$

$$\prod_{n \in \mathbb{N}} A_n = \{ f : \mathbb{N} \to \bigcup_{n \in \mathbb{N}} A_n : \forall n \in \mathbb{N},\ f(n) \in A_n \}$$

cf: $\{ (a_1, a_2, \dots) : \forall n \in \mathbb{N},\ a_n \in A_n \}$

3) A more complicated example:

$I = \mathcal{P}(\mathbb{R}) \setminus \{\phi\}$ (the set of all non-empty subsets of $\mathbb{R}$)

$\forall S \in I$ define $A_S = S$. Then

$$\prod_{S \in I} A_S = \{ f : I \to \bigcup_{S \in I} A_S : \forall S \in I,\ f(S) \in S \}.$$

Question: How do you even know that there is a function like this?

* The non-emptiness of the Cartesian product, for arbitrary collections $\{A_i\}_{i \in I}$ of non-empty sets, is equivalent to the Axiom of Choice. *